ELASTIC CONDUCTING SHELLS IN ALTERNATING ELECTROMAGNETIC FIELDS

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Paper [1] employed asymptotic integration of Maxwell's equations (in the quasi-stationary approximation) and the equations of elasticity in a thin domain occupied by the shell material, to obtain the nonlinear equations of electromechanics of thin elastic shells; it was shown that they can be linearized in the solution of two basic types of problems: problems involving the determination of damping of vibrations of shells by constant magnetic fields [2], and problems of excitation of elastic vibrations of shells by variable electromagnetic fields.

In this paper, for problems of the second type, we give the equations in vector form, the corresponding initial and boundary conditions, and the limits of applicability of the equations, and we provide a description of the general technique for solving them. For the electrodynamic part of the problem, we give an expression for the total-power functional, and we formulate the orthogonality conditions for the eigensolutions of the corresponding equations; for problems involving shells in specified harmonic fields, we write out the expansion formula for the magnetic potential of the eddy currents in eigensolutions.

As an example, we consider the problem of relative rotation of a thin-walled elastic sphere and a constant oblique magnetic field, considered in the rigid-shell approximation in [3,4]; we also obtain formulas for determining the magnetic pressure on the sphere when a constant magnetic field is applied.

1. In investigating the dynamics of thin elastic nonmagnetic shells of finite electrical conductivity in alternating electromagnetic fields, we will employ the following equations as our starting-point:

- the equation of electrodynamics of thin shells:

$$\begin{gathered}\gamma\Delta_s F+\dot{f}_n=\dot{B}_n \quad (\text{on } S), \quad \Delta\Phi=0 \quad (\text{in } V)\\ [(\operatorname{grad}\Phi)_s^+-(\operatorname{grad}\Phi)_s^-]\cdot n=0, \quad \gamma=(h\mu_0\sigma)^{-1}\\ F=\Phi_s^+-\Phi_s^-, \quad f={}^1/_2[(\operatorname{grad}\Phi)_s^++(\operatorname{grad}\Phi)_s^-]\end{gathered} \tag{1.1}$$

- the formulas for determining the magnetic pressure:

$$\mathbf{X}=-\mu_0^{-1}(\operatorname{grad}_s F\cdot \mathbf{n})\times(\mathbf{B}+\mathbf{f}) \tag{1.2}$$

- the dynamic equations of thin elastic shells:

$$Eh\mathbf{L}\mathbf{u}+\rho h\ddot{\mathbf{u}}=\mathbf{X} \tag{1.3}$$

Here S is the center surface of the shell (n is the unit normal to it); V is a simply connected infinite domain (the whole space except for S); B is the value of the magnetic induction vector on S, obtained with no shell present; Φ is the magnetic potential; f is the magnetic induction of the eddy currents in the shell; u is the displacement vector of the center surface of the shell; h, σ, E, ρ are the thickness, electrical conductivity, Young's modulus, and density of the shell; μ_0 is the magnetic constant; Δ and Δ_s are Laplace operators in V and on S; L is the operator of shell theory [5]; and the dot denotes derivatives with respect to time t.

Boundary conditions must be added to Eqs. (1.1)-(1.3). For (1.1) we have the condition that Φ be bounded at infinity, and on the edge G of the shell we have one of the following conditions: $F = 0$ on an insulated edge or $\partial F/\partial n = 0$ (here $\mathbf{n}$ is the normal to G on S) on an edge that is in contact with an ideal conductor. The ordinary boundary conditions of the theory of elastic shells [6] must be added to (1.3).

In problems of nonstationary electrodynamics, Eqs. (1.1)-(1.3) must be solved either with homogeneous initial conditions ($\Phi_0 = u_0 = u_0^{\cdot} = 0$), or with appropriate inhomogeneous initial conditions. When field $\mathbf{B}$ is switched on or off in step fashion (with $\partial/\partial t > (h^2\mu_0\sigma)^{-1}$ on the front), the initial conditions for (1.1) have the form $f_{n0} = -B_n$ and $f_{n0} = B_n$ respectively. In this case, generally speaking, Eqs. (1.3) can be employed only to determine a particular solution, whereas in determining eigensolutions (required for satisfaction of the initial conditions on u and $u^{\cdot}$) it is necessary to proceed from the equations of vibrations of shells in constant magnetic fields [1], whose eigensolutions coincide approximately with the eigensolutions of (1.3) only in the case of relatively weak magnetic fields.

Equations (1.1)-(1.3) were obtained in [1] by linearizing the general system of nonlinear equations of electromechanics of thin elastic shells. That paper considered the degree of accuracy of the equations, and demonstrated that they can be used to investigate processes in which $\partial/\partial x \ll h^{-1}$, $\partial/\partial t \ll \min(\omega_e, \omega_c)$, $\omega_e = (h^2\mu_0\sigma)^{-1}$, $\omega_c = h^{-1}\sqrt{E/\rho}$ (x is the spatial coordinate), i.e., when the dynamic theory of elastic shells is valid [5,6] and there is no skin effect.

The linear density $\mathbf{J}$, the electric field $\mathbf{e}$ of the eddy currents in the shell, and also the magnetic induction $\mathbf{b}$ generated by them in the medium can be obtained in terms of the quantities introduced via the formulas $\mathbf{J} = \mu_0^{-1}\,\mathrm{grad}_s F \times \mathbf{n}$, $\mathbf{e} = (h\sigma)^{-1}\mathbf{J}$, $\mathbf{b} = \mathrm{grad}\,\Phi$.

2. We will solve the problem in three stages; in the first stage, by integrating Eqs. (1.1), we solve the electrodynamic problem of determining the potential Φ. The magnetic pressure X is determined in the second stage by direct operations involving formula (1.2). The third stage involves dynamic calculations for an elastic shell for known pressure, on the basis of ordinary equations (1.3).

The first and third stages are the most complicated mathematically; the third is complicated mainly because of its unwieldiness. Therefore we will briefly discuss the properties of the solutions that define the first stage of Eqs. (1.1) under the boundary conditions formulated.

Homogeneous problem (1.1) ($B_n^{\cdot} = 0$), generally speaking, has eigensolutions that can be represented in the form $\Phi_k(x)\exp(\omega_k t)$, $\omega_k < 0$. The condition for them to be orthogonal is expressed by one of the equations

$$P_{t,q} = 0, \quad Q_{t,q} = 0, \quad q \neq t$$

$$P_{t,q} = \gamma\mu_0^{-1}\int_s (\mathrm{grad}_s F_t)(\mathrm{grad}_s F_q)\,ds,$$

$$Q_{t,q} = \mu_0^{-1}\int_v (\mathrm{grad}\,\Phi_t)(\mathrm{grad}\,\Phi_q)\,dv$$

When the shell is acted upon by a harmonic field with normal component (on S) $B_n(x,t) = B_n(x)\exp(i\omega t)$, the solution of inhomogeneous problem (1.1) can be obtained by expansion in eigensolutions:

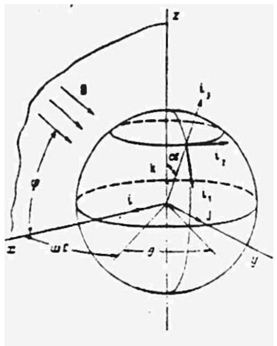
$$\Phi = \sum_{k=1}^{\infty} A_k\Phi_k(x)\exp(i\omega t), \quad A_k = i\omega(i\omega - \omega_k)^{-1}(\mu_0 Q_{k,k})^{-1}\int_s B_n(x)F_k\,ds \tag{2.1}$$

where the volume integral $Q_{k,k}$ can be replaced by a surface integral in accordance with the equality $Q_{k,k} = -P_{k,k}/\omega_k$, which follows from the reciprocity theorem (not given here) for eigensolutions of Eqs. (1.1).

The total power of the electromagnetic field of the eddy currents, defined by (1.1), is represented by the functional

$$N = \gamma\mu_0^{-1}\int_s (\mathrm{grad}_s F)^2\,ds + (2\mu_0)^{-1}\left[\int_v (\mathrm{grad}\,\Phi)^2\,dv\right]^{\cdot} + \mu_0^{-1}\int_s B_n^{\cdot} F\,ds$$

which implies (with allowance for orthogonality) that, when the shell is acted upon by



a harmonic field, the total power of heat released in the shell as a result of eddy-current heating is

$$\gamma\mu_0^{-1}\int_s (\mathrm{grad}_s F)\overline{(\mathrm{grad}_s F)}\,ds$$

3. As an example, let us consider a problem of rotation of a spherical shell of radius R (see the accompanying figure) in a constant magnetic field B whose components in the spherical coordinate system $(\alpha_1=\alpha,\ \alpha_2=\theta,\ \alpha_3=r)$, associated with the shell, are

$$\begin{gathered} B_1=-B_x\cos\alpha\cos(\omega t+\theta)+B_z\sin\alpha \\ B_2=B_x\sin(\omega t+\theta) \\ B_3=-B_x\sin\alpha\cos(\omega t+\theta)-B_z\cos\alpha \\ B_x=B\cos\varphi,\quad B_z=B\sin\varphi \end{gathered} \tag{3.1}$$

The solution of (1.1) can be obtained in the form of the expansion

$$\Phi=\sum_{j=0}^{\infty}\sum_{k=0}^{j}A_{jk}\Phi_{jk}\exp(i\omega t)$$

Allowing for the fact that the eigensolutions Φ_{jk} and eigenvalues ω_{jk} of Eqs. (1.1) can be expressed in terms of the spherical functions $Y_j^k(\alpha,\theta)$ via the formulas

$$\begin{gathered} \Phi_{jk}=\begin{Bmatrix} j(r/R)^{-(j+1)} \\ -(j+1)(r/R)^j r \end{Bmatrix} Y_j^k(\alpha,\theta),\quad \begin{Bmatrix} >R+0 \\ <R-0 \end{Bmatrix} \\ \omega_{jk}=-(2j+1)\gamma/R \end{gathered} \tag{3.2}$$

and $B_n^{\cdot}=B_3^{\cdot}=\omega B_x\sin\alpha\sin(\omega t+\theta)$, we obtain (either via a formula analogous to (2.1) or by direct substitution into (1.1))

$$\begin{gathered} \Phi={}^1/_2 RB_x\Phi_{11},\quad Y_1{}^1(\alpha,\theta)=\sin\alpha\cos(\omega t+\theta-\varkappa) \\ \varkappa=-\mathrm{arctg}\,\psi,\quad \psi=-\omega_{11}/\omega=3\gamma/(\omega R) \end{gathered} \tag{3.3}$$

Substituting (3.1)-(3.3) into (1.2) and changing over to the fixed coordinate system $(\alpha_1{}'=\alpha,\ \alpha_2{}'=\theta_1=\omega t+\theta,\ \alpha_3{}'=r)$, associated with field B, we obtain a formula for determining the magnetic pressure:

$$\begin{gathered} \mathbf{X}=\sum_{j=1}^{3}X_j\mathbf{i}_j \\ X_{1,2}=\mp\frac{3B_x}{2\mu_0}\cos\varkappa[-B_z\cos\alpha+B_x(\cos(\theta_1-x)-\cos\theta_1)\sin\alpha]\times \\ \times\begin{Bmatrix} \cos\alpha\cos(\theta_1-x) \\ \sin(\theta_1-x) \end{Bmatrix} \\ X_3=\frac{3B_x}{2\mu_0}\cos\varkappa\{[B_x\sin\alpha+B_z\cos\alpha({}^1/_2\cos(\theta_1-\varkappa)\cos\varkappa-\cos\theta_1)]\times \\ \times\cos\alpha\cos(\theta_1-\varkappa)-[B_x\sin\theta_1-{}^1/_2B_z\sin(\theta_1-\varkappa)\cos\varkappa]\sin(\theta_1-\varkappa)\} \end{gathered} \tag{3.4}$$

In (3.4) we can change over to the coordinate system associated with the shell (α_j) by using the reverse change of variable $\theta_1 = \omega t + \theta$.

The principal vector of the moment of the magnetic pressure on the shell can be obtained by integration over the surface:

$$dM = [-(X_1 \sin\theta_1 + X_2 \cos\alpha \cos\theta_1)\mathbf{i} + (X_1 \cos\theta_1 - X_2 \cos\alpha \sin\theta_1)\mathbf{j} + (X_2 \sin\alpha)\mathbf{k}]R\,ds$$
$$ds = R^2 \sin\alpha\, d\alpha\, d\theta_1$$

It is given by the formula

$$\mathbf{M} = \frac{B^2R^3}{\mu_0}\frac{1}{1+\psi^2}[\pi \sin 2\varphi(\psi\mathbf{i}+\mathbf{j}) - 2\psi\cos^2\varphi\mathbf{k}] \qquad (3.5)$$

which implies that the moment acting on the rotational axis, $\sqrt{M_x^2+M_y^2} = (\pi B^2R^3/\mu_0)\sin 2\varphi$ is independent of the angular velocity (which appears in ψ), while the retarding torque M_z has maximum value for $\omega = 3\gamma/R$ (for $\omega = 1$).

Result (3.5) coincides with that obtained by passage to the limit (for the case of small thickness) from the three-dimensional solution for a shell [3] between concentric spheres, and also with that obtained on the basis of the integrodifferential equations of [4], analogous to (1.1).*

Let us consider the determination of the elastic vibrations of a shell in a rotating lateral (ϕ = 0) magnetic field. We write Eqs. (1.3) in the adequate form [1]; for a sphere they are

$$\sum_{\eta=1}^{3}(aK_{\xi\eta}+M_{\xi\eta})v_\eta - \frac{\rho R^2}{E}\Delta_s v_\xi^{\cdot\cdot} = z_\xi \quad (\xi=1,2)$$
$$\sum_{\eta=1}^{3}(aK_{3\eta}+M_{3\eta})v_\eta + \frac{\rho R^2}{E}v_3^{\cdot\cdot} = z_3, \quad a = \frac{h^2}{12R^2} \qquad (3.6)$$

where the nonzero operators, whose degree of accuracy coincides with that of (1.1)-(1.3), have the form (ν is Poisson's ratio):

$$M_{11} = (1-\nu^2)^{-1}\Delta_s^2 + (1+\nu)^{-1}\Delta_s, \quad M_{22} = (1+\nu)^{-1}(1/2\Delta_s^2+\Delta_s)$$
$$M_{33} = 2(1-\nu)^{-1}, \quad M_{13} = M_{31} = -(1-\nu)^{-1}\Delta_s, \quad K_{33} = (1-\nu^2)^{-1}\Delta_s^2$$

where z_η (η = 1, 2, 3) are the components of the vector

$$\mathbf{z} = R^2/[(Eh)T^*\mathbf{X}] \qquad (3.7)$$

Here v_η (η = 1, 2, 3) are the components of the unknown vector **v**, in terms of which the displacement vector **u** can be expressed via the formula

$$\mathbf{u} = T\mathbf{v} \qquad (3.8)$$

The Laplace operator Δ_s and matrix operator T (T* is the formal conjugate to operator T) have the form

$$\Delta_s = \sin^{-1}\alpha\left[\frac{\partial}{\partial\alpha}\sin\alpha\frac{\partial}{\partial\alpha} + \sin^{-1}\alpha\frac{\partial^2}{\partial\theta^2}\right]$$
$$T = \begin{Vmatrix} \partial/\partial\alpha & \sin^{-1}\alpha\,\partial/\partial\theta & 0 \\ \sin^{-1}\alpha\,\partial/\partial\theta & -\partial/\partial\theta & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

In accordance with (3.4), by substituting vector **X** ($\varphi = 0$, $\theta_1 = \omega t + \theta$) into (3.7), we can bring the components z_η of vector **z** to the following form:

$$z_\eta = {}^2/_3 c\{B_{\eta 00} + B_{\eta 20}P_2^0 + B_{\eta 22}P_2^2\cos[2(\omega t+\theta-\varkappa)] + B'_{\eta 22}P_2^2\sin[2(\omega t+\theta-\varkappa)]\} \quad (\eta=1,3), \quad z_2 = -4cP_1^0 \qquad (3.9)$$

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*D. G. Vasil'ev and I. V. Simonov, "Asymptotic estimates for the complex vibrational frequencies of shells in a fluid," Preprint No. 186, Institute of Problems of Mechanics AS USSR, Moscow, 1981.

where $B_{110}=B_{120}=B_{122}=0$, $B'_{122}=1$, $B_{300}=-4d$, $B_{320}=2d$, $B'_{322}=1$, $P_n^m=P_n^m(\cos\alpha)$ is a Legendre polynomial, $d=(2\psi)^{-1}$, $c=3(BR)^2/(8Eh\mu_0)\psi/(1+\psi^2)$.

We will seek the solution of (3.6)-(3.9) in the form

$$v_\eta=A_{\eta00}+A_{\eta20}P_2^0+A_{\eta22}P_2^2\cos[2(\omega t+\theta-\varkappa)]+$$
$$+A'_{\eta22}P_2^2\sin[2(\omega t+\theta-\varkappa)]\quad(\eta=1,3),\quad v_2=A_2P_1^0 \tag{3.10}$$

Substituting (3.10) into (3.6) and (3.9), and equating coefficients for equal variables, we obtain

$$A_{100}=0,\quad A_2=-{}^1/_2c(1+\nu),\quad A_{\eta2m}^{(i)}=\delta_{\eta m}^{(i)}/\delta_m\quad(\eta=1,3,\ m=0,2)$$
$$\delta_m=(a_{11}-\lambda_m)(a_{22}-\lambda_m)-a_{12}a_{21}$$
$$\delta_{1m}^{(i)}={}^2/_3c[\chi^{-1}(a_{22}-\lambda_m)B_{12m}^{(i)}-a_{12}B_{32m}^{(i)}]$$
$$\delta_{3m}^{(i)}={}^2/_3c[(a_{11}-\lambda_m)B_{32m}^{(i)}-\chi^{-1}a_{21}B_{12m}^{(i)}] \tag{3.11}$$
$$a_{11}=\chi-(1-\nu),\quad a_{22}=a\chi^2+2(1+\nu),\quad a_{21}=\chi(1+\nu),\quad a_{22}=1+\nu$$
$$\chi=6,\quad \lambda_m=m^2(1-\nu)^2\rho\omega^2R^2/E$$

4. Let us briefly consider the approach to the solution of another problem, namely the problem of switching on a constant lateral ($\phi = 0$) magnetic field B that is stationary relative to the shell ($\omega = 0$), whose components in the system associated with the shell can be expressed by formula (3.1) for $\omega=\varphi=0$, $B_1=0$, $B_2=B$.

The magnetic potential can be obtained in the form

$$\Phi=\sum_{j=0}^{\infty}\sum_{k=0}^{j}A_{jk}\Phi_{jk}\exp(\omega_{jk}t) \tag{4.1}$$

Substituting (4.1) and (3.2) into the initial condition (see § 1) $f_{ns}=(\partial\Phi/\partial r)_s|_{t=0}=-B_3$, we obtain the solution (formulas (3.2) are given for Φ_{11} and ω_{11})

$$\Phi=-{}^1/_2Rb\Phi_{11}\exp(\omega_{11}t),\quad Y_1^1(\alpha,\theta)=\sin\alpha\cos\theta \tag{4.2}$$

Substituting (4.2) and (3.2) into (1.2), we obtain the magnetic pressure in form (3.4), where

$$X_{1,2}=\mp\frac{3B^2}{2\mu_0}\beta(t)(\beta(t)-1)\sin\alpha\cos\theta\begin{Bmatrix}\cos\alpha\ \cos\theta\\ \sin\alpha\end{Bmatrix}$$
$$X_3=\frac{3B^2}{2\mu_0}\beta(t)({}^3/_2\beta(t)-1)(\cos^2\alpha\cos^2\theta+\sin\alpha\sin\theta) \tag{4.3}$$
$$\beta(t)=\exp(-3\gamma t/R)$$

Analysis of (4.3) shows that at the initial instant the tangential components of the pressure are equal to zero ($X_1 = X_2 = 0$); then they initially increase, reaching a maximum for $t={}^1/_3R\ln 2/\gamma$, then decrease, whereas the normal pressure X_3 is nonzero and has maximum value at the initial instant.

Determination of a particular solution of the dynamic equations of a spherical elastic shell (3.6), (3.7) for specified pressure (4.3) does not entail any fundamental difficulties, and will not be considered here.

By substituting (3.10) and (3.11) into (3.8), we obtain

$$u_1=(-{}^1/_2A_{120}+3A_{122}\cos\Omega+3A'_{122}\sin\Omega)\sin 2\alpha$$
$$u_2=(A_{210}-{}^3/_2A_{122}\sin\Omega+{}^3/_2A'_{122}\cos\Omega)\sin\alpha$$
$$u_3=(A_{300}+{}^1/_4A_{320})+{}^3/_4A_{320}\cos 2\alpha+{}^3/_2(A_{322}\cos\Omega+A'_{322}\sin\Omega)(1-\cos 2\alpha),$$
$$\Omega=2(\omega t+\theta-\varkappa)$$

Resonances appear when the angular velocity ω is equal to half the circular frequency corresponding to natural oscillations of an elastic shell following the law $P_2^2(\cos\alpha)\exp(2i\theta)$. (Here $\delta_2=0$, A_{322}, A'_{322} become infinite.)

The first term of expression (3.9) for z_3 determines the constant external pressure $B^2/[2\mu_0(1+\psi^2)]$ on the shell. For a copper shell ($\sigma = 10^7$ $(\Omega\cdot m)^{-1}$) with $h = 10^{-3}$ m, $R = 10^{-1}$ m in a field B = 1 T, it is expressed as a function of ω by the formula $4\cdot 10^5(1+(2,4\cdot 10^3/\omega)^2)^{-1}$ $N\cdot m^{-2}$; as B increases, it increases in proportion to its square. For the given shell, $\omega_e = 8\cdot 10^4$ sec^{-1}, $\omega_s = 3.6\cdot 10^6$ sec^{-1} and the formulas obtained are valid for $\omega < \omega_e$ (see §1).

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