DAMPING OF OSCILLATIONS OF SHELLS BY WEAK MAGNETIC FIELDS

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The problem of free oscillations of thin elastic shells in a vacuum (s-problem) [1] has a real spectrum of eigenvalues, that defines oscillations without attenuation in time. The problem of oscillations of shells made of electrically conducting materials in a stationary magnetic field (sm-problem) [2] has a complex spectrum, and the corresponding oscillations attenuate in time. In this paper we investigate the sm-problem and we determine the effect of weak magnetic fields (the concept of "weak" is made more specific) on the solutions of the s-problem. Our emphasis is on the estimation of the imaginary parts of the eigenvalues, which govern the attenuation of oscillations in the sm-problem. All the general results of the paper are equally valid for shells and plates.

1. In considering the sm-problem, we will employ the following equations as a starting-point, with the assumption that the oscillations obey an $\exp(i\omega\tau)$ law (τ is time):

$$L_{lj}u_j+\lambda^2u_l=\beta B_jN_{lj}(F) \tag{1.1}$$

$$\begin{gathered}\gamma\Delta_s F+i\lambda f=i\lambda(A_1A_2)^{-1}\partial/\partial\alpha_k A_q(B_k u_3-B_3u_k)\ (\text{on } S)\\ \Delta\Phi=0\ (\text{in } V),\ F=(\Phi)_s^+-(\Phi)_s^-,\ f=(H_3^{-1}\partial\Phi/\partial\alpha_3)_s\\ \beta=(2Eh\mu_0)^{-1},\ \gamma=(2h\mu_0\sigma)^{-1}\sqrt{\rho/E},\ \lambda=\omega\sqrt{\rho/E},\ i=\sqrt{-1}\end{gathered} \tag{1.2}$$

Here j, l, k, q are summation indexes ($l, q=1, 2, 3$; $k, q=1, 2$; $k\neq q$); ($\alpha_1, \alpha_2, \alpha_3$) are parameters of a triorthogonal coordinate system specified in space V surrounding the shell, in which the center surface S of the shell is assumed to lie on the coordinate surface $\alpha_3 = 0$; $\mathbf{B}(B_1, B_2, B_3)$ is the value of the magnetic induction vector on S (assumed to be known from the solution of the magnetostatic problem; its components are assumed to be not identically zero); $\mathbf{u}(u_1, u_2, u_3)$ is the displacement vector of the center surface of the shell; Φ is the potential function in terms of which the perturbed component $\mathbf{b}$ of the magnetic induction in the medium surrounding the shell is defined ($\mathbf{b}=\operatorname{grad}\Phi$, $f=(b_3)_s$); the subscript s means that the values of the corresponding quantities are taken on the surface S (for $\alpha_3 = 0$), where $(\ )_s^{\pm}=(\ )_{\alpha_3=\pm0}$, Δ and Δ_s are Laplace operators in V and on S respectively; μ_0 is the magnetic constant; h, σ, ρ, E are the half-thickness, conductivity, density, and Young's modulus of the shell material; H_j are the Lame coefficients ($A_j=(H_j)_s$), L_{lj} are the operators of shell theory (given, e.g., in [3]); and N_{lj} are differential operators that are introduced for notational convenience; they are equal to zero except for $N_{k3}=-N_{3k}=$ $=-A_k^{-1}\partial/\partial\alpha_k$ ($k=1, 2$).

Equations (1.1)-(1.2) were obtained from [2] with accuracy $O(\eta^{1-p})$ (η is the relative half-thickness of the shell; p is the index of variability of the desired state [3]). To these we should add homogeneous conditions of attachment of the shell edges and the condition that the solution be bounded at infinity.

2. We will assume that the magnetic field ensures smallness of the parameter

$$\varepsilon=\max_s|\mathbf{B}|/E_*^{1/2},\quad E_*=2E\mu_0 \tag{2.1}$$

We will seek the solution of Eqs. (1.1)-(1.2) in the form of series:

$$\Phi=E_*^{1/2}\varepsilon\sum_{n=0}^{\infty}\varepsilon^{2n}\Phi_n\ [F,f];\quad \lambda=\sum_{n=0}^{\infty}\varepsilon^{2n}\lambda_n\ [u_1,u_2,u_3] \tag{2.2}$$

For the functions in brackets, the series can be constructed in accordance with the expressions directly in front of them.

Substituting (2.2) into (1.1)-(1.2) and equating coefficients for equal powers of ε, we obtain equations which can be reduced to the following form:

$$L_{ij}u_{j0}+\lambda_0^2 u_{i0}=0 \tag{2.3}$$

$$L_{ij}u_{jn}+\lambda^2 u_{in}=-\sum_{\substack{m+l+g=n\\ g\neq n}}\lambda_m\lambda_l u_{ig}+h^{-1}B_{0j}N_{ij}(F_{n-1}) \qquad (n=1,2,\ldots) \tag{2.4}$$

$$\begin{gathered}\gamma\Delta_s F_n+i\lambda_0 f_n=-i\sum_{\substack{m+g=n\\ g\neq n}}\lambda_m f_g+i\sum_{m+g=n}\frac{\lambda_m}{A_1A_2}\frac{\partial}{\partial\alpha_k}A_q(B_{0k}u_{3g}-B_{03}u_{kg})\\ \Delta\Phi_n=0,\ F_n=(\Phi_n)_s^+-(\Phi_n)_s^-,\ f_n=(H_3^{-1}\partial\Phi_n/\partial\alpha_3)_s\ (n=0,1,2,\ldots)\\ B_{0j}=B_j/\max_s|\mathbf{B}|\end{gathered} \tag{2.5}$$

Equations (2.3)-(2.5) enable us to construct a process of successive determination of all the unknown functions in expansions (2.2). First let us consider the equations of the s-problem (2.3). Assume that $\{\lambda_0, u_{j0}\}$ is an ensemble consisting of the value of the frequency parameter and the form (or mode), belonging to some eigensolution of (2.3). Then Eqs. (2.5), corresponding to n = 0, define an m-problem of integration of the Laplace equation in V with an inhomogeneous mixed boundary condition on mathematical cut S, whose solution yields the function F_0. (Equations (2.4) and the first equation in (2.5) are written in such a way that, here and henceforth, the quantitites on their right sides are always known from the preceding treatment.)

Going to (2.4) (n = 1), we obtain the s-problem of forced oscillations of the shell at the resonant frequency. As we know, this problem can have a bounded solution only if the forcing forces do not perform work on the displacements, defined by the corresponding eigensolution. This means that the right sides of (2.4) should be orthogonal, in a certain sense, to eigenform $\{u_{j0}\}$. Following [1,4], and assuming that the solution of (2.4) is sought in the form of an expansion in eigensolutions of Eqs. (2.3), we obtain functions $\{u_{j1}\}$ and the existence condition for a bounded solution, from which we can express the value of λ_1.

Our further treatment of Eqs. (2.3)-(2.5) involves successive solution of the problems formulated above for Eqs. (2.5) and (2.4).

The formulas for the expansion terms λ, obtained from the orthogonality condition, have the form

$$\begin{gathered}\lambda_n=\frac{1}{2\lambda_0 I_{j0,j0}}\iint_s\Big[-\sum_{\substack{m+l+g=n\\ m,l,g\neq n}}\lambda_m\lambda_l u_{ig}+h^{-1}B_{0j}N_{ij}(F_{n-1})\Big]u_{i0}\,ds\\ I_{m,n}=\iint_s u_m u_n\,ds,\qquad ds=A_1A_2\,d\alpha_1\,d\alpha_2\qquad(n=1,2,\ldots)\end{gathered} \tag{2.6}$$

3. Using the results of §2, we can formulate the problem of approximate determination of the eigenvalues λ of Eqs. (1.1)-(1.2) with an accuracy corresponding to retention of the first two expansion terms in (2.2). They can be obtained using the formula

$$\lambda=\Lambda+\frac{1}{2\Lambda I_{j,j}}\iint_s\beta B_j N_{ij}(F)u_i\,ds \tag{3.1}$$

where Λ, u_j is the eigensolution of the s-problem for the equations

$$L_{ij}u_j+\Lambda^2 u_i=0 \tag{3.2}$$

while F is the solution of the m-problem for the equations

$$\begin{gathered}\gamma\Delta_s F+i\Lambda f=i\Lambda(A_1A_2)^{-1}\partial/\partial\alpha_k A_q(B_k u_3-B_3u_k)\\ \Delta\Phi=0,\quad F=(\Phi)_s^+-(\Phi)_s^-,\quad f=(H_3^{-1}\partial\Phi/\partial\alpha_3)_s\end{gathered} \tag{3.3}$$

As we can see from (3.3), function F is complex, and therefore the second term in (3.1) comprises a complex correction to the real value Λ of the s-problem.

4. As shown in §2, the complex correction $\delta\Lambda$ to Λ $(\lambda=\Lambda+\delta\Lambda)$ in (3.1) is small when the strength of the stationary magnetic field is relatively modest. In a number of practical problems, therefore, it is advisable to confine oneself to estimating the imaginary part λ.

For this purpose we employ the asymptotic method [1,3,5], based on the fact that the parameter of the relative half-thickness $\eta = h/R$ is small (R is the characteristic dimension of the shell).

We write the complex quantities in (3.1)-(3.3) in the following form:

$$\delta\Lambda=\delta\Lambda_1+i\delta\Lambda_2[\Phi, F, f], \quad \delta\Lambda_1=\mathrm{Re}\,\delta\Lambda, \ \delta\Lambda_2=\mathrm{Im}\,\delta\Lambda \tag{4.1}$$

Substituting representations (4.1) of functions Φ, F, f into (3.3) and separating the real and imaginary parts of the equations, we obtain a system with real coefficients:

$$\begin{gathered}\gamma\Delta_* F_2+\Lambda f_1=\Lambda(A_1A_2)^{-1}\partial/\partial\alpha_k A_q(B_k u_3-B_3 u_k), \ \gamma\Delta_* F_1-\Lambda f_2=0\\ \Delta\Phi_n=0, \ F_n=(\Phi_n)_s^+-(\Phi_n)_s^-, \ f_n=(H_3^{-1}\partial\Phi_n/\partial\alpha_3).\end{gathered} \tag{4.2}$$

We dilate the scales and change the unknowns and coefficients of the equations in accordance with the expressions

$$\begin{gathered}\alpha_j=R\eta^p\xi_j, \ R\Lambda=\eta^r\Lambda_*, \ R\delta\Lambda_k=\eta^{r_k}\delta\Lambda_k^\circ\\ \Phi_k=E_*^{1/2}R\eta^{t_k}\Phi_k^\circ, \ F_k=E_*^{1/2}R\eta^{t_k}F_k^\circ, \ f_k=E_*^{1/2}\eta^{t_k-p}f_k^\circ\\ u_j=R\eta^{d_j}u_j^\circ, \ \gamma=\eta^{a-1}\gamma_0, \ B_j=E_*^{1/2}\eta^{-g}B_j^\circ\end{gathered} \tag{4.3}$$

All the numbers in the exponents of η are chosen in such a way that for the state under consideration, which is a solution of (3.1), (3.2), (4.1), (4.2), for a shell with specified electromechanical properties, the quantities on the right sides of (4.3) indicated by degree signs are of order $O(\eta^\circ)$, while differentiation of the unknowns with respect to ξ_j does not cause them to change asympototically. This means, in particular, that p comprises an index of variability; r (which coincides, to within the sign, with what was adopted in [1]) defines the asymptotic behavior of the principal value of Λ, while r_k is the correction (r_1 and r_2 of the real $\delta\Lambda_1$ and imaginary $\delta\Lambda_2$ respectively) to the eigenvalue of the s-problem; $\eta^{-g} = \varepsilon$, where ε is given by formula (2.1).

Substituting (4.3) into (4.2) and equating the exponents of the asymptotically principal terms (which must necessarily include some of the terms containing u_j), we obtain the following formulas for determining t_1 and t_2:

$$\begin{gathered}t_1=\begin{cases}2\varkappa+g-d, & \varkappa>0\\ -g+d, & \varkappa\leqslant 0\end{cases} \quad t_2=\begin{cases}\varkappa-g+d, & \varkappa>0\\ -\varkappa-g+d, & \varkappa\leqslant 0\end{cases}\\ \varkappa=p+r+1-a, \ d=\min(d_j)\end{gathered} \tag{4.4}$$

Making substitutions (4.1) and (4.3) in (3.1), separating the real and imaginary parts of the resultant expression, bounding the integrals on the right sides, and equating the exponents of η of the expressions on different sides of the equals sign, we obtain formulas for r_1 and r_2:

$$r_1=\begin{cases}-2a+1+p+r-2g, & \varkappa>0\\ -1-p-r-2g, & \varkappa\leqslant 0\end{cases} \quad r_2=\begin{cases}-a-2g, & \varkappa>0\\ a-2(1+p+r)-2g, & \varkappa\leqslant 0\end{cases} \tag{4.5}$$

Formulas (4.5) and the corresponding expressions (4.3) define the asymptotic form of the real and imaginary parts of the correction to the eigenvalues of the s-problem, that appear as a result of weak magnetic fields. If the correction to the eigenvalues Λ of the s-problem is asymptotically small as compared to Λ, we will assume that the magnetic fields are "weak." The smallness condition is expressed by the inequality $\min(r_1, r_2)>r$ and reduces to a constraint on the parameter g, that determines (together with (4.3)) the strength level of the stationary magnetic field. Thus, weak magnetic fields can be considered to be those for which

$$|\mathbf{B}|<B^*=\eta^{-g}E_*^{1/2}, \quad g=\begin{cases}-(a+r)/2, & \varkappa>0\\ -(1+p)/2-r, & \varkappa\leqslant 0\end{cases} \tag{4.6}$$

With such fields, it is legitimate to employ the method and formulas in §§2, 3. The effect of stronger fields on the modes and frequencies of free oscillations of the shell becomes decisive, and the process described in §2 diverges.

It follows from (4.6) that B^* increases with the oscillation frequency (r decreases), i.e., B^* is smallest at the first frequencies, which are characterized, according to [1], by parameter values $p = 0$, $r = 0$ for a shell of arbitrary curvature, and by values $p = 0$, $r = 1$ for a plate. Then for a shell and plate of stainless steel with characteristic dimensions $R = 1$ m, $2h = 2$ mm ($\eta = 10^{-3}$, $a = 1.7$), the first frequencies fall in the regions $\kappa < 0$ and $\kappa > 0$ respectively, while the B^* values calculated on the basis of (4.6) amount to 24 T (for the shell) and $7 \cdot 10^{-2}$ T (for the plate). This means, in particular, that the oscillations of the primary spectrum of such a shell can practically always be calculated using the method of §2.

Remark. The spectra of shells of zero and negative curvature contain lower frequencies. For them, B^* will be smaller. These cases can be analyzed using formulas (4.6).

5. The asymptotic results of §4 yield the following conclusions. Within the scope of applicability of two-dimensional shell theory [1] and the hypotheses of magnetoelasticity [2], defined by the inequalities $0 \leqslant p < 1$, $\max(-1, a-2) < r < \infty$, we can identify two regions in which the effect of weak magnetic fields on free oscillations is findamentally different: a low-frequency region, defined by the inequality $p+r > a-1$ $(\varkappa > 0)$, and a high-frequency region in which $p+r < a-1$ $(\varkappa < 0)$.

In the low-frequency region, the imaginary part of the correction to the eigenvalues of the s-problem is asymptotically greater than the real part $(\delta\Lambda_1 < \delta\Lambda_2)$, and has the following order:

$$\delta\Lambda_2 = R^{-1} O(\eta^{-a-2g}) \tag{5.1}$$

As can be seen, it is independent of both the frequency and the variability of the corresponding eigensolutions (associated with the asymptotic parameters r and p), but depends on the parameters a and g defined by the last expressions in (4.3), whose substitution into (5.1) yields $\delta\Lambda_2 \sim \sigma B^2/\sqrt{\rho E}$.

Thus, the imaginary correction to the eigenvalue is proportional to the conductivity of the shell and to the square of the magnetic field strength.

The eigenvalue of Eqs. (1.1)-(1.2) in the low-frequency region can be determined approximately from the formula

$$\omega = \Omega + i\sigma B^2 \rho^{-1} A \tag{5.2}$$

where Ω is the eigenvalue of the s-problem, while the multiplier $A = O(\eta^0)$.

In the high-frequency region, the imaginary part of the correction is always less than the real part $(\delta\Lambda_1 > \delta\Lambda_2)$, and has the following order:

$$\delta\Lambda_2 = R^{-1} O(\eta^{a-2(1+p+r)-2g}) \tag{5.3}$$

It is inversely proportional to the conductivity of the shell, proportional to the square of the magnetic field strength, and depends on the frequency and variability of the desired unknown eigensolution.

Using the asymptotic properties of the eigensolutions of the s-problem [1], we obtain the following estimates for $\delta\Lambda_2$ in the high-frequency region for three fundamental modes of oscillation of the shell:

- for quasi-transverse oscillations with low variability

$$r=0, \quad \delta\Lambda_2 = R^{-1} O(\eta^{a-2(1+p)-2g}), \quad 0 \leqslant p < 1/2$$

- for quasi-transverse oscillations with large variability

$$r=1-2p, \quad \delta\Lambda_2 = R^{-1} O(\eta^{a-2+p-2g}), \quad 1/2 \leqslant p < 1$$

- for quasi-tangential oscillations

$$r = -p, \quad \delta\Lambda_2 = R^{-1}O(\eta^{a-2-2p}), \quad 0 < p < 1$$

In the high-frequency region, with increasing oscillation number (i.e., with increasing p), the imaginary correction to the eigenvalue for quasi-transverse oscillations initiall increases sharply ($\sim\eta^{-2p}$), and then gradually decreases ($\sim\eta^{p}$); for quasi-tangential oscillations $\delta\Lambda_2$ does not depend asymptotically on the oscillation number.

As follows from (4.5), in the high-frequency region the real correction to the eigenvalue is independent of the conductivity of the shell. (The expression for r_1 does not contain the parameter a.)

In the high-frequency region the imaginary corrections to the eigenvalues are asymptotically smaller than in the low-frequency region (by a factor of $\eta^{-2\kappa}$). Coupled with the fact that the principal real part Λ of the eigenvalues is greater in the high-frequency region than in the low-frequency one, this means that at high frequencies the damping of oscillations by magnetic fields is generally negligible.

In the intermediate region, where $p + r = a - 1$ $(\varkappa = 0)$, the real and imaginary parts of the correction to the eigenvalue are asymptotically equal $(\delta\Lambda_1 \sim \delta\Lambda_2)$.

Study [2] offers a solution of the sm-problem of axisymmetric free oscillations of an infinite cylindrical shell in a longitudinal magnetic field; a frequency equation is obtained which can be reduced (within the scope of applicability of two-dimensional shell theory, for states with variability corresponding to p > 0) tothe form $\omega^2 - i\omega\alpha(1 + i\omega\delta^{-1})^{-1} - \Omega^2 = 0$, where $\delta = K/(\mu_0\sigma h)$, $\alpha = \sigma B^2/\rho$, K is the wave number, and Ω is the frequency of the corresponding oscillations in the s-problem.

For weak magnetic fields, this equation admits the following approximate solutions (written to within the first term containing i):

- in the low-frequency region (parameters Ω/δ, α/Ω small):

$$\omega = \Omega + i\,{}^1/_2\alpha \qquad (5.4)$$

- in the high-frequency region (parameters δ/Ω, α/Ω small):

$$\omega = \Omega + {}^1/_2\alpha\delta\Omega^{-1} + i\,{}^1/_2\alpha\delta^2\Omega^{-2} \qquad (5.5)$$

Formulas (5.4) and (5.5) fully confirm the qualitative conclusions deriving from an analysis of (5.1) and (5.3); moreover, formula (5.4) coincides with (5.2) for A = 0.5.

In the low-frequency region, the example we have considered is also in agreement with the results of [6].

The possibility in principle of employing the above approaches in the solution of the magnetoelasticity problem agrees with the results of [7]. The parameters Ω/δ and α/Ω, used as the basis for the parameters, can be regarded as analogs of the magnetic Reynolds number and magnetic pressure [7] in shell theory.

We should note that in this study, as in [2,6,7], the issue of damping of elastic oscillations was considered exclusively from the standpoint of the effect of magnetic fields on them, without allowance for internal friction, which is a primary factor in damping in a number of cases. Our proposed method can also be extended to the case of equations that allow for the effect of internal friction (e.g., by introducing complex elastic moduli [8]) in addition to magnetic fields; however, this is beyond the scope of this paper.

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